ANOSOV FLOWS WITH GIBBS MEASURES ARE ALSO BERNOULLIAN†

BY

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ABSTRACT

We consider a special flow S^t over a shift in the space of sequences (X, μ) constructed using a continuous f with

$$\sum_{n=0}^{\infty} n \cdot \sup_{x,y:x_i=y_i,|i|\leqq n} |f(x)-f(y)| < \infty.$$

We formulate a condition for μ such that the K-flow S^t is a B-flow.

1.

A geometric idea of Ornstein and Weiss [8] turns out to be applicable to transitive Anosov flows of class C^2 on compact Riemannian manifolds, relative to a wide class of K-measures which Sinai considered in [11] and called Gibbs measures (see also [3]). This class includes the so-called smooth measures, which induce conditional measures equivalent to normalized Riemannian volume on the elements of a measurable partition consisting of subsets of contracting (or expanding) leaves. For geodesic flows on compact manifolds of negative curvature, these measures coincide with invariant Riemannian volume. Sinai's construction of Gibbs measures [11] made essential use of a special representation of an Anosov flow $\{T^t\}$ on M, obtained with the aid of a Markov partition (see [10], [2], [9]). This partition determines a matrix $\Pi = \{\pi_{ij}\}$, $\pi_{ij} = 0, 1$, of order r, with the property that for some $s > 0$ the elements of the matrix Π^s are positive. Using the matrix Π, one constructs a space $X_\Pi = X \subset \{1, 2, \cdots, r\}^Z$ of sequences $x = \{x_i\}_{i=-\infty}^{\infty}$, $\pi_{x_i,x_{i+1}} = 1$, with metric

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$$d(x',x'') = \sum_i 2^{-|i|} e(x_i', x_i''), e(x_i', x_i'') = \begin{cases} 0 & x_i' = x_i'' \\ 1 & x_i' \neq x_i''. \end{cases}$$

X_Π is compact.

In X one can define the shift automorphism ϕ: $(\phi x)_i = x_{i-1}$. The Markov partition enables one to define: (i) a positive continuous function $f(x)$ on X satisfying a Holder condition; (ii) a special flow $S^t = (\phi, f)$ in the space

$$W = (X,f) = \{(x,y): \ x \in X, \ 0 \leqq y < f(x), \ (x,f(x)) = (\phi x, 0)\}$$

with the direct product topology, so that for $t < \inf_{x \in X} f(x)$

$$S^t(x,y) = \begin{cases} (x, y+t), & t < f(x) - y \\ (\phi x, t + y - f(x)), & t \geqq f(x) - y. \end{cases}$$

S^t is uniquely defined for all other t by the condition that it be a one-parameter group of transformations; (iii) a continuous mapping $\psi: W \to M$ such that $\psi S^t = T^t \psi$. If we now let ν be an S^t-invariant Borel measure in W such that ψ fails to be bijective on a set of ν-measure zero, then the flows S^t in (W, ν) and T^t in $(M, \psi^* \nu)$ are isomorphic (for any Borel set $A \subset M$, $\psi * \nu(A) = \nu(\psi^{-1}A)$).

This was the method used by Sinai in [11] to construct invariant Gibbs K-measures for transitive Anosov flows of class C^2, carrying out the actual construction for special flows S^t in W. Sinai's Gibbs measure ν in W induces on X a ϕ-invariant Borel K-measure μ $(d\nu = (d\mu \times dt)\bar{f}^{-1}, \bar{f} = \int_X f(x) d\mu)$, possessing the property:

(1) $\lim_{n \to \infty} \mu(x_1 \mid x_0, x_{-1}, \cdots, x_{-n}) = \mu(x_1 \mid x_0, x_{-1}, \cdots) \geqq a > 0$, for every sequence $\{x_i\}_{-\infty}^0$.

(2) For all $\bar{x} = \{\bar{x}_i\}_{-\infty}^0$, $\bar{\bar{x}} = \{\bar{\bar{x}}_i\}_{-\infty}^0$, $\bar{x}_i = \bar{\bar{x}}_i \mid i \mid \leqq n$,
and certain constants $C, \kappa > 0$, $0 < \rho < 1$, we have

$$\left| \frac{\mu(x_1/\bar{x})}{\mu(x_1/\bar{\bar{x}})} - 1 \right| < C\rho^{n\kappa}. \tag{3}$$

Let α denote the partition of X into sets $x_0 = i$, $i = 1, 2, \cdots, r$, and $\alpha_s^t = \bigvee_{k=s}^t \phi^k \alpha$. By the construction of the space $X_\Pi = X$, we can define for $A \in \alpha$ and $\bar{x}, \bar{\bar{x}} \in \alpha_{-\infty}^0 / A$ a bijective mapping η: $\bar{x}$ onto $\bar{\bar{x}}$ such that $x \in \bar{x}$ and $\eta(x) = z \in \bar{\bar{x}}$ lie in the same atom of the partition α_0^∞, that is, $\{z_i\}_{-\infty}^0 = \bar{\bar{x}}$ and $z_i = x_i$, $i \geqq 0$. It clearly follows from (3) that if $\bar{x}, \bar{\bar{x}} \subset A \in \alpha_{-n}^0$ then for any measurable $C \subset \bar{x}$, $\mu(C \mid \bar{x}) > 0$

$$\left|\frac{\mu(\eta C/\bar{\bar{x}})}{\mu(C/\bar{x})} - 1\right| < \gamma(n) \tag{4}$$

where $\gamma(n) \to 0$ as $n \to \infty$.

When α is a Markov partition for an Anosov diffeomorphism, the atoms of $\alpha_{-\infty}^{0}$ are subsets of contracting leaves, while the elements of α_0^{∞} are subsets of expanding leaves (see [10], [9]). Then the mapping η (sometimes called the canonical mapping) maps one contracting subset onto another via transversal intersection with an expanding subset. For an algebraic automorphism of a torus with Haar measure μ and a geodesic flow on a manifold of negative curvature with invariant Riemannian volume, the mapping η satisfies condition (4) (see [1]).

The geometric idea of Ornstein and Weiss, applied to these cases [8], is essentially based on condition (4).

Let us say that a function f continuous on X belongs to class $\Upsilon(\Upsilon')$ if, for all $x, x' \in X$, $x_i = x_i'$ for $|i| \leqq n$,

$$|f(x) - f(x')| < \beta(n), \ \sum_{n=0}^{\infty} \beta(n) < \infty \ \left(\sum_{n=0}^{\infty} n \cdot \beta(n) < \infty\right).$$

Our main result (Theorems 3.1, 4.3) is that if condition (4) holds in (X, μ) (or even a weaker condition 2.1, see below, which does not require that X have a matrix stucture), $0 < f \in \Upsilon'$ and for fixed t, S^t is a K-automorphism in (W, v), then S^t is isomorphic to a B-shift. In particular, we obtain the following theorems.

THEOREM 1.1 *Transitive Anosov flows of class C^2 with Gibbs measures are B-flows.*

THEOREM 1.2. *If (X, μ, ϕ) is a Markov chain, $0 < f \in \Upsilon'$, and for fixed t, S^t is a K-automorphism in (W, v), then S^t is isomorphic to a B-shift.*

2.

Here we shall study condition (4). If P is a property that holds for all atoms of a partition α, with the possible exception of a set of atoms whose union has measure less than ε, we shall say that P holds for ε-almost every atom of α, or for ε-almost every atom A of α.

We now formulate a weaker version of condition (4). Let (X, μ) be a ϕ-invariant subset of $\{1, 2, \cdots r\}^Z$.

CONDITION 2.1. For any $\varepsilon > 0$, there exists a set P of atoms of $\alpha_{-\infty}^{0}$, $\mu(P) > 1 - \varepsilon$, a set Q of atoms of α_0^{∞}, $\mu(Q) > 1 - \varepsilon$, and a number $N = N(\varepsilon) > 0$, such that for all $x^0_{-N} \in \alpha^0_{-N}$, all $\bar{x}, \bar{\bar{x}} \in P \cap x^0_{-N}$, and any set $A \subset Q$ of atoms of α_0^∞, we have: $\mu(A \mid \bar{x}) > 0$ iff $\mu(A \mid \bar{\bar{x}}) > 0$ and

$$\left| \frac{\mu(A/\bar{x})}{\mu(A/\bar{\bar{x}})} \right| - 1 < \varepsilon.$$

If $X = X_\Pi$, then $A \cap \bar{x} \neq \emptyset$ iff $A \cap \bar{\bar{x}} \neq \emptyset$ and $A \cap \bar{\bar{x}} = \eta(A \cap \bar{x})$.

Henceforth, the notation $\bar{x} \in x^0_{-N}$ will imply that x^0_{-N} is a set of atoms of $\alpha^0_{-\infty}$. Moreover, it will be clear from the context whether a set C is to be considered as a set of atoms of some partition or as a subset of the space X.

PROPOSITION 2.2. *If ϕ is a K-automorphism in (X, μ) and Condition 2.1 holds, then α is a weak Bernoulli partition.*

α is said to be a weak Bernoulli partition if for any $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) > 0$ such that for all $n' \geqq n \geqq n_0$, all $m > 0$ and ε-almost every atom $B \in \alpha^0_{-m}$,

$$\sum_{A \in \alpha_n^{n'}} \left| \mu(A/B) - \mu(A) \right| < \varepsilon. \tag{5}$$

PROOF. Let Condition 2.1 hold for $\varepsilon > 0$, P, Q and N. Since ϕ is a K-automorphism and α^0_{-N} is finite, there exists $k_0 = k_0(\varepsilon)$ such that for all $k' \geqq k \geqq k_0$, ε-almost every atom $A \in \alpha_k^{k'}$ and all $x^0_{-N} \in \alpha^0_{-N}$, $\mu(x^0_{-N}) > 0$,

$$\left| \frac{\mu(x^0_{-N}/A)}{\mu(x^0_{-N})} - 1 \right| < \varepsilon.$$

That is,

$$\left| \frac{\mu(A/x^0_{-N})}{\mu(A)} - 1 \right| < \varepsilon. \tag{6}$$

Set $A = A' \cup A''$, $A' = A \cap Q$, $A'' = A \cap \bar{Q}$. Then by (6),

$$\left| \mu(A') - \mu(A'/x^0_{-N}) \right| < \varepsilon\mu(A) + \mu(A'') + \mu(A''/x^0_{-N}). \tag{7}$$

Further

$$\mu(A'/x^0_{-N}) = \mu(A'/x^0_{-N} \cap P) \cdot \mu(P/x^0_{-N}) + \mu(A'/x^0_{-N} \cap \bar{P}) \cdot \mu(\bar{P}/x^0_{-N})$$

$$\mu(A'/x^0_{-N} \cap P) = \int_{\bar{x} \in x^0_{-N} \cup P} \mu(A'/\bar{x}) d\mu_{x^0_N x^0_{-N} \cap P}(\bar{x}).$$

By Condition 2.1, for any $\bar{x} \in x_{-N} \cap P$, we have $\mu(A' | \bar{x}) > 0$ iff $\mu(A' | x^0_{-N} \cap P) > 0$ and

$$\left|\frac{\mu(A'/x^0_{-N}\cap P)}{\mu(A'|\bar{x})}-1\right|<\varepsilon.$$

Since $\mu(P) > 1-\varepsilon$ and $\mu(Q) > 1-\varepsilon$, there exists a set R of atoms of α^0_{-N}, $\mu(R) > 1-2\varepsilon^{\frac{1}{2}}$, such that for all $x^0_{-N}\in R$:

$$\mu(P/x^0_{-N}) > 1-\varepsilon^{\frac{1}{2}} \text{ and } \mu(Q/x^0_{-N}) > 1-\varepsilon^{\frac{1}{2}}.$$

Then, for $\bar{x}\in x^0_{-N}\cap P$, $x^0_{-N}\in R$,

$$|\mu(A'/x^0_{-N})-\mu(A'/\bar{x})| < 2\varepsilon^{\frac{1}{2}}\mu(A'/\bar{x})+\varepsilon^{\frac{1}{2}}\mu(A'/x^0_{-N}\cap P).$$

It then follows from (7) that for $\bar{x}\in S = R\cap P$, $\mu(S) > 1-3\varepsilon^{\frac{1}{2}}$ and $\bar{x}\in x^N_0\in R$,

$$|\mu(A'/\bar{x})-\mu(A')| < 2\varepsilon^{\frac{1}{2}}\mu(A'/\bar{x})+\varepsilon^{\frac{1}{2}}\mu(A'/x^0_{-N}\cap\bar{P})+\mu(A'')+\varepsilon\mu(A)+\mu(A''/x^0_{-N}).$$

This is true for all $k'\geqq k\geqq k_0$ and ε-almost every atom $A\in\alpha^{k'}_k$. Hence, for $\bar{x}\in S$,

$$\sum_{A\in\alpha^{k'}_k}|\mu(A/\bar{x})-\mu(A)| < 17\varepsilon^{\frac{1}{2}}.$$

Since ϕ is a K-automorphism, there exists $m_0 = m_0(\varepsilon) > 0$ such that for all $m'\geqq m\geqq m_0$ and ε-almost every atom $B\in\alpha^{-m}_{-m'}$,

$$\left|\frac{\mu(S/B)}{\mu(S)}-1\right|<\varepsilon.$$

But $\mu(A/B) = \mu(A/B\cap S)\ \mu(S/B)+\mu(A/B\cap\bar{S})\ \mu(\bar{S}/B)$ and

$$\begin{aligned}\sum_{A\in\alpha^{k'}_k}|\mu(A/B)-\mu(A)| &= \sum_{A\in\alpha^{k'}_k}\Big|\mu(S/B)\int_{\bar{x}\in B\cap S}[\mu(A/\bar{x})-\mu(A)]\,d\mu_{B\cap S}(\bar{x})\\ &\quad+\mu(A)\mu(S/B)-\mu(A)+\mu(A/B\cap\bar{S})\mu(\bar{S}/B)\Big|\\ &\leqq \mu(S/B)\int_{\bar{x}\in B\cap S}\sum_{A\in\alpha^{k'}_k}|\mu(A/\bar{x})-\mu(A)|]\,d\mu_{B\cap S}(\bar{x})\\ &\quad+8\varepsilon^{\frac{1}{2}}\leqq 25\varepsilon^{\frac{1}{2}}.\end{aligned}$$

This clearly implies (5), if we set $n_0 = k_0(\varepsilon^2/17)+m_0(\varepsilon^2/8)$. ▌

Remark. The proof remains valid without change if Condition 2.1 is weakened as follows: for any collection $\{A_i\}$ of sets $A_i\subset Q$ of atoms of α^∞_0, $A_i\cap A_j=\varnothing$, $(i\neq j)$,

$$\sum|\mu(A_i/\bar{x})-\mu(A_i/\bar{\bar{x}})| < \varepsilon. \tag{8}$$

Since α is a generating partition, we obtain the following theorem from [7].

THEOREM 2.3. *If Condition* (8) *holds, the K-shift* ϕ *is Bernoullian. In particular, transitive Anosov diffeomorphisms are Bernoullian relative to Gibbs measures* (see [4], [11]).

3.

In this section we prove Theorem 3.1.

THEOREM 3.1. *If Condition* 2.1 *holds in* (X,μ), $0 < f^* \leqq f \in \Upsilon$ *is bounded and constant on atoms of* α_0^∞ *and for fixed* t, S^t *is a K-automorphism in* (W,ν), *then* S^t *is a B-shift.*

Our proof will employ the concept of very weak Bernoulli (VWB) partitions, defined in [7], [8].

Let (W,B,ν) be a measurable space, S an invertible measure-preserving transformation, and γ a finite partition. The following two theorems may be found in [7], [6], ([8].

THEOREM A. *If* γ *is* VWB *then* $(W, \bigvee_{-\infty}^{\infty} S^n\gamma, \nu, S)$ *is a B-shift.*

Here $\bigvee_{-\infty}^{\infty} S^n\gamma$ denotes the smallest σ-algebra with respect to which all the $S^i\gamma$ are measurable.

THEOREM B. *If* $A_1 \subseteq A_2 \subseteq \cdots$ *is an increasing sequence of S-invariant* σ*-algebras,* $V_1^\infty A_i = B$, *and for each* i (W,A_i,ν,S) *is a B-shift, then* (W,B,ν,S) *is a B-shift.*

In order to formulate sufficient conditions for a VWB partition, we need the following concepts. Let (X_1, m_1) and (X_2, m_2) be two measurable spaces. A mapping $\Theta: X_1 \to X_2$ is said to be ε measure preserving if there is a set $E_1 \subset X_1$, $m_1(E_1) \leqq \varepsilon$, such that for all $A \subset X_1 - E_1$, $m_1(A) > 0$,

$$\left| m_2(\Theta A)/m_1(A) - 1 \right| \leqq \varepsilon.$$

Let $\{\alpha_i\}_1^m$ be a sequence of partitions in X. Then the $\{\alpha_i\}_1^m$-name of $x \in X$ is the sequence $l_i = l_i(x)$ determined by $x \in A_{l_i}^{(i)}$, $\alpha_i = \{A_1^{(i)}, A_2^{(i)}, \cdots, A_{a_i}^{(i)}\}$.

Let γ be a finite partition in (W,B,ν,S) and $\gamma_p^q = \bigvee_{i=p}^q S^i\gamma$. We shall consider $A \in \gamma_p^q$ as a measurable space with σ-algebra B/A and normalized conditional measure $\nu \mid A$, and also the sequences of partitions $\{S^{-i}\gamma\}_1^m$ in (W,B,ν) with name-function $l_i(w)$ and $\{S^{-i}\gamma \mid A\}_1^m$ in $(A, B \mid A, \nu \mid A)$ with name-function $m_i(y)$. Let e be defined on the integers by $e(j) = 1$ for $j \neq 0$ and $e(0) = 0$.

The following theorem is proved in [8].

THEOREM C. *Suppose that for any $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) > 0$ such that for all $n' \geqq n \geqq n_0$, ε-almost every atom $A \in \gamma_n^{n'}$ and any $m \geqq 1$ one can construct an ε measure-preserving-mapping θ: (W, B, ν) onto $(A, B|A, \nu|A)$ such that*

$$(9) \qquad \begin{cases} \dfrac{1}{m} \sum\limits_{i=1}^{m} e(l_i(w) - m_i(\theta w)) \leqq \varepsilon, & w \in W - E \\ \nu(E) \leqq \varepsilon. \end{cases}$$

Then γ is VWB.

Since f is constant on atoms of α_0^∞ there exists a partition ξ of W into atoms $\{S^y x_0^\infty, 0 \leqq y < f(x_0^\infty), x_0^\infty \in \alpha_0^\infty\}$.

Let $x_{-n}^n \in \alpha_{-n}^n$ and $2/n \leqq y + 1/n \leqq \inf_{x \in x_{-n}^n} f(x)$. Define an n-cube in W as a set

$$w_n = \bigcup_{t=y-1/n}^{y+1/n} S^t x_{-n}^n .$$

Let ζ be the partition of w_n into sets $\{C^t = S^t x_{-\infty}^n, t \in [y - 1/n, y + 1/n], x_{-\infty}^n \subset x_{-n}^n\}$ and g the partition of the same set into sets $\{\bigcup_{t=y-1/n}^{y+1/n} S^t x_{-\infty}^n, x_{-\infty}^n \subset x_{-n}^n\}$. We write $\partial w_n = S^{y+1/n} x_{-n}^n \cup S^{y-1/n} x_{-n}^n$. It is convenient to introduce the projection $\pi: W \to X$, defined by $\pi(w) = x$ for $w = (x,y)$, $0 \leqq y < f(x)$.

Now suppose that Condition 2.1 holds for $\varepsilon > 0$, $P, Q, N(\varepsilon) > 0$, and let $n \geqq N(\varepsilon)$. Set $P_n = \phi^n P$, $Q_n = \phi^n Q$, $P_n' = \bigcup_{x \in P_n} \bigcup_{t=0}^{f(x)} S^t x$, $Q_n' = \bigcup_{x \in Q_n} \bigcup_{t=0}^{f(x)} S^t x$. P and Q are, of course, treated here as subsets of X. It is clear that $\nu(P_n') \geqq 1 - \tau\varepsilon$ and $\nu(Q_n') \geqq 1 - \tau\varepsilon$, where $\tau = \sup_{x \in X} f(x)/\bar{f}$. The set P_n consists of atoms of $\alpha_{-\infty}^n$, and since f is constant on atoms of α_0^∞ it follows that Q_n' consists of atoms of ξ.

Let w_n be some n-cube in (W, ν) and g' the set of all atoms G of $g \mid w_n$ such that $G \subset P_n'$. Then Condition 2.1 implies the following.

PROPOSITION 3.2. *For any $\varepsilon > 0$, $n \geqq N(\varepsilon)$, any set $A \subset Q_n'$ of atoms of ξ and any $\bar{G}$, $\bar{\bar{G}} \in g'$, we have: $\nu(A \mid \bar{G}) > 0$ iff $\nu(A \mid \bar{\bar{G}}) > 0$, and*

$$(10) \qquad \left| \frac{\nu(A/\bar{G})}{\nu(A/\bar{\bar{G}})} - 1 \right| < \varepsilon.$$

PROOF. By the definition of ν,

$$\nu(A/\bar{G}) = \left[\int_{y-1/n}^{y+1/n} \mu(\pi(A \cap \bar{C}_\zeta^t)/\bar{x}_{-\infty}^n) dt \right] n/2$$

where $\bar{x}_{-\infty}^n = \pi\bar{G}$ and $\bar{C}_\zeta^t \in \zeta \mid \bar{G}$.

A similar expression is valid for $\bar{\bar{G}}$, $\bar{\bar{x}}^{n}_{-\infty} = \pi\bar{\bar{G}}$ and $\bar{\bar{C}}^{t}_{\zeta} = \zeta \mid \bar{\bar{G}}$. Now, if A^{t} is a set of atoms $C_{\xi} \in A$ such that $C_{\xi} \cap S^{t}x^{n}_{-n} \neq \emptyset$, then $\pi A^{t} \subset Q_{n}$ consists of atoms of α_0^{∞} and

$$\pi(A \cap \bar{C}^{t}_{\zeta}) = \pi A^{t} \cap \bar{x}^{n}_{-\infty} \text{ and } \pi(A \cap \bar{\bar{C}}^{t}_{\zeta}) = \pi A^{t} \cap \bar{\bar{x}}^{n}_{-\infty}.$$

Then, by Condition 2.1, $\mu(\pi(A \cap \bar{C}^{t}_{\zeta}) \mid \bar{x}^{n}_{-\infty}) > 0$ iff $\mu(\pi(A \cap \bar{\bar{C}}^{t}_{\zeta}) \mid \bar{\bar{x}}^{n}_{-\infty}) > 0$ and

$$\left| \frac{\mu(\pi(A \cap \bar{C}^{t}_{\xi})/\bar{x}^{n}_{-\infty})}{\mu(\pi(A \cap \bar{\bar{C}}^{t}_{\zeta})/\bar{\bar{x}}^{n}_{-\infty})} - 1 \right| < \varepsilon.$$

This clearly implies (10). ∎

Now set $\tilde{w}_n = w_n \cap P'_n \cap Q'_n$ and

$$\tilde{g} = \bar{g}/w_n = \{\tilde{G} = G \cap \tilde{w}_n,\ G \in g',\ \nu(\tilde{G}/G) > 0\}.$$

It is clear from (10) that for $\bar{G}$, $\bar{\bar{G}} \in \tilde{g}$ and a set $A \subset Q'_n$ of atoms of ξ

$$\left| \frac{\nu(A/\bar{G})}{\nu(A/\bar{\bar{G}})} - 1 \right| < 2\varepsilon. \tag{11}$$

Now let A be a set of atoms of ξ such that $\nu(A \cap G/G) > 0$ for all $G \in \tilde{g}$. Let θ_G be some measure-preserving mapping of $(G, \nu/G)$ onto $(A_G = A \cap G, \nu/A_G)$ and $\theta = \theta_{\tilde{w}_n}$ a mapping of $(\tilde{w}_n, \nu/\tilde{w}_n)$ onto $(\tilde{A} = A \cap \tilde{w}_n, \nu/\tilde{A})$ such that $\theta/G = \theta_G$ for $G \in \tilde{g}$.

LEMMA 3.3. *θ is a 4ε-measure preserving mapping. If $z \in \tilde{w}_n$ then πz, $\pi(\theta z) \in x^{n}_{-\infty}$. For all $t \geqq 0$ $d_W(S^{t}z, S^{t}\theta z) \leqq 3/n + r_n$, where d_W is the metric in W and $r_n = \sum_{i=n}^{\infty}\beta(i)$. (See the definition of $f \in \Upsilon$.)*

PROOF. The second assertion follows from the construction of the mapping θ_G. We have

$$d_W(S^{t}z, S^{t}\theta z) \leqq 3/n + \left| \sum_{i=0}^{l(t,z)-1} (f(\phi^{i}\pi z) - f(\phi^{i}\pi\theta z)) \right| \leqq 3/n + \sum_{i=n}^{\infty} \beta(i),$$

where $l(t, z)$ is the number of times the trajectory $S^{t}(\pi z, 0)$ hits X during time t.

To prove the first assertion, let v be the partition of $\tilde{A}$ into atoms $\{V_G = G \cap \tilde{A}$, $G \in \tilde{g}\}$ and $H \subset \tilde{A}$. We have

$$\nu(H/\tilde{A}) = \int_{V \in v} \nu(H/V) d\nu_{\tilde{A}}(V). \tag{12}$$

Let F be a set of atoms of v. Then

$$\nu(F/\tilde{A}) = \int_{G\in\tilde{g}:G\cap F\neq\phi} \nu(V_G/G)d\nu_{\tilde{w}}(G) \Big/ \int_{G\in\tilde{g}} \nu(V_G/G)d\nu_{\tilde{w}}(G).$$

It follows from (11) that for $\bar{G}, \bar{\bar{G}} \in \tilde{g}$,

$$\left|\frac{\nu(V_{\bar{G}}/\bar{G})}{\nu(V_{\bar{\bar{G}}}/\bar{\bar{G}})} - 1\right| < 2\varepsilon.$$

Therefore,

$$\left|\frac{\nu(F/\tilde{A})}{\nu(\theta^{-1}F/\tilde{W})} - 1\right| < 4\varepsilon. \tag{13}$$

Since $\nu(H/V_G) = \nu(\theta^{-1}H/G)$, it follows from (12) and (13) that

$$\left|\frac{\nu(H/\tilde{A})}{\nu(\theta^{-1}H/\tilde{w})} - 1\right| < 4\varepsilon.$$

This proves the lemma. ∎

The continuity of f easily implies the following.

LEMMA 3.4. *For any $\varepsilon > 0$, there exist $n > 0$ and a partition of W, $\beta_n = \{B_0, B_1, \cdots, B_{b_n}\}$, such that $\nu(B_0) \leqq \varepsilon$ and B_i, $i = 1, \cdots, b_n$, are n-cubes in (W, ν).*

LEMMA 3.5. *There exists an ascending sequence of partitions $\gamma_1 < \gamma_2 < \cdots$ in (W, ν) such that $\bigvee_{i=1}^{\infty}\gamma_i$ is the partition into points and for every $i \geqq i_0$ we have $\gamma_i = \{\Gamma_0^i, \Gamma_1^i, \cdots, \Gamma_{l_i}^i\}$, where Γ_k^i, $k = 1, \cdots, l_i$, are i-cubes in (W, ν).*

PROOF OF THEOREM 3.1. Set $S^t = S$ for $t > 0$; we claim that for any fixed $i \geqq i_0$ the partition $\gamma = \gamma_i$ of Lemma 3.5 is VWB in (W, ν, S). Our assertion will then follow from Theorems A and B.

Set $\partial\gamma = \bigcup_{\Gamma\in\gamma}\partial\Gamma$. It is clear that $\nu(\partial\gamma) = 0$. Let $O_k(\partial\gamma) = \bigcup_{t=-(3/k+r_k)}^{3/k+r_k} S^t(\partial\gamma)$. Let $\varepsilon > 0$ be given and suppose that $\nu(O_K(\partial\gamma)) < \varepsilon^2$ for $K = K(\varepsilon)$. Let $n \geqq \max(K(\varepsilon), N(\varepsilon))$ and let $\beta = \beta_n = \{B_0^n, \cdots, B_{b_n}^n\}$, $\nu(B_0^n) \leqq \varepsilon$, be the partition of Lemma 3.4. For each n-cube $B_j^n, j = 1, \cdots, b_n$, we consider $\tilde{B}_j^n = B_j^n \cap P_n' \cap Q_n'$ $\tilde{B} = \bigcup_{j=1}^{b_n}\tilde{B}_j^n$, $\nu(\tilde{B}) \geqq 1 - (2\tau + 1)\varepsilon$.

Since S is K-automorphism, there exists $M_0 = M_0(\varepsilon) \geqq i \sup_{x\in X} f(x)$ such that for all $M' \geqq M \geqq M_0$, ε-almost every atom $A \in \bigvee_{k=M}^{M'} S^k\gamma$ and all $\tilde{B}_j^n$, $j = 1, \cdots, b_n$, $\nu(\tilde{B}_j^n) > 0$, we have

$$\left|\frac{\nu(\tilde{B}_j^n/A)}{\nu(\tilde{B}_j^n)} - 1\right| < \varepsilon. \tag{14}$$

Let θ_j be the 4ε measure-preserving mapping of Lemma 3.3 of $(\tilde{B}_j^n, \nu/\tilde{B}_j^n)$ onto $(\tilde{A}_j = A \cap \tilde{B}_j^n, \nu/\tilde{A}_j)$. We define a mapping θ of (W, ν) onto $(A, \nu/A)$ by $\theta = \theta_j/\tilde{B}_j^n$, $j = 1, \cdots, b_n$, with $\theta/W - \tilde{B}$ an arbitrary mapping onto $A - \tilde{A}$ where $\tilde{A} = \bigcup_{j=1}^{b_n} \tilde{A}_j$. It follows from (14) that θ is a $q\varepsilon$ measure-preserving mapping, where $q = \max(5, 2\tau + 1)$.

For $m \geqq 1$, consider the sequence of partitions $\{S^{-k}\gamma\}_1^m$ in (W, ν) with name-function $l_k(w)$ and $\{S^{-k}\gamma \,|\, A\}_1^m$ in $(A, \nu/A)$ with name-function $m_k(z)$. We claim that θ satisfies the conditions of Theorem C.

Since $n > i$, it follows that $\pi\Gamma$ consists of atoms of $\alpha_{-\infty}^n$ for each atom $\Gamma \in S^{-k}\gamma$, $k \geqq 0$. Hence, by Lemma 3.3, the $S^{-k}\gamma$ name of a point $w \in \tilde{B}$ and the $S^{-k}\gamma$ name of the point $\theta w \in \tilde{A}$ can be different only if $S^k w \in O_n(\partial\gamma)$. By the choice of n, we have $\nu(O_n(\partial\gamma)) < \varepsilon^2$. Let $N(w)$ denote the number of indices k, $0 \leqq k \leqq m$, such that $S^k w \in O_n(\partial\gamma)$. Then

$$\{w \in \tilde{B}: \frac{1}{m} \sum_{j=1}^{m} e(m_j(\theta w) - l_j(w)) > \varepsilon\} \subset \left\{w \in \tilde{B}: \frac{N(w)}{m} > \varepsilon\right\}$$

$$= \left\{w \in \tilde{B}: \frac{1}{m} \sum_{j=1}^{m} r_j(w) > \varepsilon\right\} = E_m$$

where

$$r_j(w) = \begin{cases} 1, & S^j w \in O_n(\partial\gamma) \\ 0, & \text{otherwise.} \end{cases}$$

The expectation of this random variable satisfies the inequality $Er_j(w) < \varepsilon^2$, and by Chebyshev's inequality $\nu(E_m) < \varepsilon$. This implies (9), if we set $E = E_m \cup (W - \tilde{B})$, $\nu(E) < (2\tau + 2)\varepsilon$. ∎

REMARK. The proof remains valid almost without change if the integrable function $0 < f^* \leqq f \in \Upsilon$ is not bounded on X.

4.

In this section we do not suppose that f is constant on the atoms of α_0^∞.

In order to carry our result over to this case, we need the following concept. Let $f, g \in L_u^1(X)$ and $\bar{f} = \bar{g}$. We shall say that f is homologous to g ($f \sim g$) relative ϕ if there exists a measurable function $u(x)$ on (X, μ) such that almost everywhere $f(x) = g(x) + u(x) - u(\phi^{-1}x)$ (see [5], [11]). For example, $f(x)$ and $g(x) = f(\phi^{-k}x) = \phi^k f(x)$ are homologous, with $u(x) = \sum_{i=0}^{k-1} \phi^i f(x)$. The relation $\sim$ is reflexive, symmetric and transitive.

The proof of the following lemma may be found in Gurevic [5].

LEMMA 4.1. *If $f \sim g$, then the special flows (ϕ, f) and (ϕ, g) constructed over (X, μ) with the aid of the functions f and g are isomorphic.*

LEMMA 4.2. (See also [11].) *If the function $0 < f^* \leqq f \in \Upsilon'$ is bounded on X, then there exists a function $0 < g^* \leqq g \in \Upsilon$, homologous to f, which is bounded and constant on the atoms of α_0^∞.*

PROOF. For $x \in X$, $n = 1, 2, \cdots$ and $x \in x_{-n}^n \in \alpha_{-n}^n$, we set

$$f_n(x) = \int_{x_{-n}^n} f(z) d\mu_{x_{-n}^n}(z)$$

where the integration is performed with respect to the measure μ / x_{-n}^n. All the functions f_n are in Υ', bounded, constant on the atoms of α_{-n}^n, and $f_n \geqq f^* > 0$, $n = 1, 2, \cdots$. Set $h_n(x) = f_n(x) - f_{n-1}(x)$. For all $x \in X$, $|h_n(x)| \leqq 2\beta(n-1)$.

Suppose that $2 \sum_{k=m+1}^\infty \beta(k-1) \leqq f^*/2$ for $m > 0$. We have

$$f(x) = f_m(x) + \sum_{i=m+1}^\infty h_i(x). \tag{15}$$

The series in (15) converges uniformly. Each function h_i is constant on the atoms of α_{-i}^i and the function $\phi^i h_i(x) = h_i(\phi^{-i} x)$ is constant on the atoms of α_0^{2i}. Consider

$$g'(x) = f_m(x) + \sum_{i=m+1}^\infty \phi^i h_i(x). \tag{16}$$

The series in (16) converges uniformly and the function g' is continuous, bounded and constant on the atoms of α_{-m}^∞. By our choice of m, $g' \geqq f^*/2 = g^* > 0$. We show that $g' \in \Upsilon$. Let $y, z \in x_{-n}^n$, $n \geqq m$. Since $\phi^i h_i$ is constant on the atoms of α_0^{2i}, $\phi^i h_i(y) = \phi^i h_i(z)$ for $i \leqq n/2$ and

$$|g'(y) - g'(z)| = \Big| \sum_{i=[n/2]+1}^\infty (\phi^i h_i(y) - \phi^i h_i(z)) \Big| \leqq 4 \sum_{i=[n/2]}^\infty \beta(i) = \gamma_n.$$

Since $\sum n\beta(n) < \infty$, $\sum \gamma_n < \infty$ and $g' \in \Upsilon$.

Consider

$$u(x) = \sum_{i=m+1}^\infty \sum_{k=0}^{i-1} \phi^k h_i(x). \tag{17}$$

Since $\sum_{i=0}^\infty i\beta(i) < \infty$, the series in (17) converges uniformly. It is readily seen that $f(x) = g'(x) + u(x) - \phi u(x)$, that is, $f \sim g'$. The function $g = \phi^m g'$ is in Υ, bounded, constant on atoms of α_0^∞, and $g \geqq g^* > 0$. Moreover, $g \sim g'$ and so $g \sim f$.

We obtain the following.

THEOREM 4.3. *If Condition 2.1 holds in (X, μ), $0 < f^* \leqq f \in \Upsilon'$, and for fixed t, S^t is a K-automorphism then S^t is a B-shift.*

THEOREM 4.4. *Transitive Anosov flows of class C^2 with Gibbs measures are B-flows.*

THEOREM 4.5. *If (X, μ, ϕ) is a Markov chain, $0 < f \in \Upsilon'$, and S^t is a K-flow then S^t is a B-flow.*

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